

CONVERGENCE RATES OF THE SPLITTING SCHEME FOR PARABOLIC LINEAR STOCHASTIC CAUCHY PROBLEMS

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Abstract. We study the splitting scheme associated with the linear stochastic Cauchy problem

$$\begin{aligned} dU(t) &= AU(t) dt + dW(t), \quad t \in [0, T], \\ U(0) &= x, \end{aligned}$$

Here A is the generator of an analytic C_0 -semigroup $S = \{S(t)\}_{t \geq 0}$ on a Banach space E and $W = \{W(t)\}_{t \geq 0}$ is a Brownian motion with values in a fractional domain space E_β associated with A . We prove that if $\alpha, \beta, \gamma, \theta \geq 0$ are such that $\gamma + \theta < 1$ and $(\alpha - \beta + \theta)^+ + \gamma < \frac{1}{2}$, then the approximate solutions $U^{(n)}$ converge to the solution U in the space $C^\gamma([0, T]; E_\alpha)$, both in L^p -means and almost surely, with rate $1/n^\theta$.

Key words. Splitting scheme, stochastic evolution equations, analytic semigroups, γ -radonifying operators, γ -boundedness, stochastic convolutions, Lie-Trotter product formula.

AMS subject classifications. Primary: 35R15, 60H15; Secondary: 47D06, 60J35

1. Introduction. We are concerned with the convergence of the splitting scheme for the stochastic linear Cauchy problem

$$\begin{cases} dU(t) = AU(t) dt + dW(t), & t \in [0, T], \\ U(0) = x, \end{cases} \quad (\text{SCP}_x)$$

where A is the generator of a C_0 -semigroup $S = \{S(t)\}_{t \geq 0}$ on a real Banach space E , $W = \{W(t)\}_{t \geq 0}$ is an E -valued Brownian motion on a probability space (Ω, \mathbb{P}) , and $x \in E$ is an initial value which is kept fixed throughout the paper. The concept of the scheme is to alternately add an increment of the Brownian motion W and run the semigroup S on a time interval of equal length. Taking time steps $\Delta t^{(n)} := T/n$ and writing $t_j^{(n)} := jT/n$ and $\Delta W_j^{(n)} := W(t_j^{(n)}) - W(t_{j-1}^{(n)})$, this generates a finite sequence $\{U_x^{(n)}(t_j^{(n)})\}_{j=0}^n$ defined by

$$\begin{aligned} U_x^{(n)}(t_0^{(n)}) &:= x, \\ U_x^{(n)}(t_j^{(n)}) &:= S(\Delta t^{(n)})(U_x^{(n)}(t_{j-1}^{(n)}) + \Delta W_j^{(n)}), \quad j = 1, \dots, n. \end{aligned}$$

We have the explicit formula

$$U_x^{(n)}(t_j^{(n)}) = S(t_j^{(n)})x + \sum_{i=1}^j S(t_{j-i+1}^{(n)})\Delta W_i^{(n)}, \quad j = 0, \dots, n.$$

Assuming the existence of a unique solution U_x of the problem (SCP_x) (see Proposition 3.2 below), we may ask for conditions ensuring the convergence of $U_x^{(n)}(T)$ to $U_x(T)$ in $L^p(\Omega; E)$ for some (all) $1 \leq p < \infty$ or even in E almost surely. In order to describe our

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approach we start by noting that each $U_x^{(n)}(t_j^{(n)})$ can be represented as a stochastic integral of the discretised function

$$S^{(n)}(t) := \sum_{j=0}^n 1_{I_j^{(n)}} \otimes S(t_j^{(n)}), \quad t \in [0, T],$$

where $I_0^{(n)} = \{0\}$ and $I_j^{(n)} = (t_{j-1}^{(n)}, t_j^{(n)}]$ for $j = 1, \dots, n$. Indeed, defining the stochastic integral of a step function in the obvious way, we have

$$U_x^{(n)}(t_j^{(n)}) = S^{(n)}(t_j^{(n)})x + \int_0^{t_j^{(n)}} S^{(n)}(t_j^{(n)} - s) dW(s), \quad j = 0, \dots, n. \quad (1.1)$$

On the other hand, the exact solution of (SCP_x) , if it exists, is given by the stochastic convolution integral

$$U_x(t) := S(t)x + \int_0^t S(t-s) dW(s), \quad t \in [0, T]. \quad (1.2)$$

For the precise definition of the stochastic integral we refer to Section 3. Comparing (1.1) and (1.2) we see that the problem of convergence of the splitting scheme is really a problem of convergence of ‘Riemann sums’ for stochastic integrals. Let us henceforth put

$$U_x^{(n)}(t) := S^{(n)}(t)x + \int_0^t S^{(n)}(t-s) dW(s), \quad t \in [0, T].$$

The second formula interpolates the data in the identity (1.1) in a way that makes them easily accessible with continuous time techniques; other possible interpolations, such as piecewise linear interpolation, do not have this advantage. Needless to say, in Theorems 1.1 and 1.2 below we are primarily interested in what happens at the time points $t = t_j^{(n)}$. From $S^{(n)}(t_j^{(n)})x = S(t_j^{(n)})x$ we see that

$$U_x^{(n)}(t_j^{(n)}) - U_x(t_j^{(n)}) = U_0^{(n)}(t_j^{(n)}) - U_0(t_j^{(n)})$$

for all $x \in E$, and therefore it suffices to analyse convergence of the splitting scheme with initial value 0. In what follows, in order to simplify notations we shall write $U(t) := U_0(t)$ and $U^{(n)}(t) := U_0^{(n)}(t)$.

Our first result extends and simplifies previous work by Kühnemund and the second-named author [21, Theorems 4.3 and 5.2].

THEOREM 1.1. *Each of the conditions (a) and (b) below guarantees that the problem*

$$\begin{cases} dU(t) = AU(t) dt + dW(t), & t \in [0, T], \\ U(0) = 0, \end{cases} \quad (SCP_0)$$

admits a unique solution $U = \{U(t)\}_{t \in [0, T]}$ which satisfies

$$\lim_{n \rightarrow \infty} \left(\sup_{t \in [0, T]} \mathbb{E} \|U^{(n)}(t) - U(t)\|^p \right) = 0$$

for all $1 \leq p < \infty$:

- (a) E has type 2;
- (b) S restricts to a C_0 -semigroup on the reproducing kernel Hilbert space associated with W .

The class of spaces satisfying condition (a) includes all Hilbert spaces and the spaces $L^p(\mu)$ for $2 \leq p < \infty$. It follows from the results in [26] that condition (b) is satisfied if the transition semigroup associated with the solution process is analytic.

The main result of this article, Theorem 1.2 below, concerns actual convergence rates for the splitting scheme in the case that the semigroup S is *analytic* on E . The convergence is considered in suitable Hölder norms in space and time, with explicit bounds for the convergence rate.

We denote by E_α the fractional power space of exponent $\alpha \geq 0$ associated with A (see Section 4 for more details).

THEOREM 1.2. *Suppose that the semigroup S is analytic on E and that W is a Brownian motion in E_β for some $\beta \geq 0$. Then the problem (SCP_0) admits a unique solution $U = \{U(t)\}_{t \in [0, T]}$, and for all $\alpha, \gamma, \theta \geq 0$ such that $\gamma + \theta < 1$ and $(\alpha - \beta + \theta)^+ + \gamma < \frac{1}{2}$ one has the estimate*

$$(\mathbb{E} \|U^{(n)} - U\|_{C^\gamma([0, T]; E_\alpha)}^p)^{\frac{1}{p}} \lesssim \frac{1}{n^\theta}, \quad 1 \leq p < \infty,$$

with implied constant independent of $n \geq 1$.

By a Borel-Cantelli argument, this result implies the almost sure convergence of $U^{(n)}$ to U in $C^\gamma([0, T]; E_\alpha)$ with the same rates.

The proof of Theorem 1.2 heavily relies on the theory of γ -radonifying operators and γ -boundedness techniques. Standard techniques from stochastic analysis which are commonly used in connection with the problems considered here, such as Itô's formula and the Burkholder-Davis-Gundy inequalities, are unavailable in the present general framework (unless one makes additional assumptions on E , such as martingale type 2 or the UMD property). We also cannot use factorisation techniques (as introduced by Da Prato, Kwapień and Zabczyk [7]), the reason being that the semigroup property on which this technique relies fails for the discretised semigroup $S^{(n)}$.

EXAMPLE 1.3. Theorem 1.2 may be applied to second order elliptic operators of the form

$$Af(x) = \sum_{i,j=1}^d a_{ij}(x) \partial_{ij} f(x) + \sum_{i=1}^d b_i(x) \partial_i f(x) + c(x) f(x).$$

Under minor regularity assumptions on the coefficients $a_{ij} = a_{ji}$, b_i and c , such operators generate analytic semigroups on $E = L^q(\mathbb{R}^d)$ with $1 < p < \infty$ (see [25, Chapter 3]) and one has $E_\alpha = H^{2\alpha, q}(\mathbb{R}^d)$ for all $0 < \alpha < \frac{1}{2}$. Applying Theorem 1.2 (with $\beta = 0$), we obtain convergence of the splitting scheme in the space $C^\gamma([0, T]; H^{2\alpha, q}(\mathbb{R}^d))$ for any $\gamma \geq 0$ such that $0 < \alpha + \gamma < \frac{1}{2}$. By the Sobolev embedding $H^{2\alpha, q}(\mathbb{R}^d) \hookrightarrow C_0^{2\alpha-d/q}(\mathbb{R}^d)$ [39, Section 2.8], this implies the convergence of the splitting scheme in the mixed Hölder space $C^\gamma([0, T]; C_0^{2\alpha-d/q}(\mathbb{R}^d))$. As a consequence, we obtain convergence in the mixed Hölder space

$$C^\gamma([0, T]; C_0^{2\delta}(\mathbb{R}^d)), \quad \gamma, \delta \geq 0, \quad \gamma + \delta < \frac{1}{2},$$

with rate $1/n^\theta$ for any $\theta < \frac{1}{2} - \gamma - \delta$; this rate improves when the noise is more regular. Similar results can be obtained for elliptic operators on smooth domains

$D \subseteq \mathbb{R}^d$ subject to various types of boundary conditions (as long as they generate an analytic semigroup on $L^q(D)$).

For semi-linear (Stratonovich type) SPDEs governed by second order elliptic operators on \mathbb{R}^d and driven by multiplicative noise, convergence in $E = L^2(\mathbb{R}^d)$ of splitting schemes like the one considered here has been proved by various authors [2, 3, 12, 14, 27]. Using techniques from PDE and stochastic analysis it is shown by Gyöngy and Krylov [14] that, with respect to the norm of $E = L^2(\mathbb{R}^d)$, for finite-dimensional noise and with sufficiently smooth coefficients one obtains the maximal estimate

$$(\mathbb{E} \sup_{t \in [0, T]} \|U^{(n)} - U\|_{L^2(\mathbb{R}^d)}^p)^{\frac{1}{p}} \lesssim \frac{1}{n}, \quad 1 \leq p < \infty.$$

Our result is valid in the full scale of spaces $L^q(\mathbb{R}^d)$ and infinite-dimensional noise, with a rate which (for smooth enough noise) is only slightly worse than $1/n$ and is independent of $1 \leq q < \infty$. More precisely, for $\beta \geq \frac{1}{2} + \alpha$ and taking $\gamma = 0$ we obtain uniform convergence with rate $1/n^\theta$ for any $0 \leq \theta < 1$. In addition to that we obtain Hölder regularity in both space and time. On the other hand, as we already mentioned, Gyöngy and Krylov [14] consider the semi-linear case and multiplicative noise.

The next example shows that by working in suitable fractional extrapolation spaces (this technique is explained in [10]; see also [4, 5]), the assumption that W is a Brownian motion can be weakened to W being a cylindrical Brownian motion (see, e.g., [30, 32] for the definition).

EXAMPLE 1.4. The stochastic heat equation on the unit interval $[0, 1]$ with Dirichlet boundary conditions driven by space-time white noise can be put into the present framework by taking for E the extrapolation space F_ρ with $F = L^q(0, 1)$ and $\rho < -\frac{1}{4}$. As we shall explain in Example 4.12, this entails the convergence of the splitting scheme in the mixed Hölder space

$$C^\gamma([0, T]; C_0^{2\delta}[0, 1]), \quad \gamma, \delta \geq 0, \quad \gamma + \delta < \frac{1}{4},$$

with rate $1/n^\theta$ for any $\theta < \frac{1}{4} - \gamma - \delta$.

It is shown in [8] that any approximation scheme for a one-dimensional stochastic heat equation with additive space-time white noise which incorporates the contributions of the noise only by means of the terms ΔW_k , $k = 1, \dots, n$, cannot have a convergence rate better than $1/n^{\frac{1}{4}}$. This shows that the exponent $\frac{1}{4}$ in Example 1.4 is the best possible.

The field of numerical approximation of stochastic partial differential equations (SPDEs) is a very active one; an up-to-date overview of the available results can be found in [19]. In [13] convergence rates are considered for various approximations schemes in space and time of a quasi-linear parabolic SPDE driven by white noise. The authors obtain a convergence rate $1/n^{\frac{1}{4}}$ in L^p for an implicit Euler scheme. In [36] convergence in probability is proved (without rates) for the same SPDE with state-dependent dispersion. Rates for path-wise convergence are given for quasi-linear parabolic SPDEs in [15, 18, 24], albeit only for colored noise. It seems likely that the methods of this paper can be extended to the implicit Euler scheme and to semi-linear problems with multiplicative noise; we plan to address such extensions in a future paper.

The paper is organised as follows. Section 2 presents some preliminary material about spaces of γ -radonifying operators. The proofs of Theorems 1.1 and 1.2 are presented in Sections 3 (Theorems 3.4, 3.5) and 4 (Theorem 4.9), respectively.

It is known that each of the conditions in Theorems 1.1 and 1.2 implies that the solution process U has continuous trajectories. In the final Section 5 we present an example which shows that without any additional assumptions on the space E and/or the semigroup S the splitting scheme may fail to converge even if a solution U with continuous trajectories exists.

2. Preliminaries. Let $\{\gamma_j\}_{j \geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space (Ω, \mathbb{P}) , let \mathcal{H} be a real Hilbert space (later we shall take $\mathcal{H} = L^2(0, T; H)$, where H is another real Hilbert space) and E a real Banach space. A bounded operator R from \mathcal{H} to E is called γ -*summing* if

$$\|R\|_{\gamma_\infty(\mathcal{H}, E)}^2 := \sup_h \mathbb{E} \left\| \sum_{j=1}^k \gamma_j R h_j \right\|^2,$$

is finite, where the supremum is taken over all finite orthonormal systems $h = \{h_j\}_{j=1}^k$ in \mathcal{H} . It can be shown that $\|\cdot\|_{\gamma_\infty(\mathcal{H}, E)}$ is indeed a norm which turns the space of γ -summing operators into a Banach space.

The space $\gamma(\mathcal{H}, E)$ of γ -*radonifying* operators is defined to be the closure of the finite rank operators under the norm $\|\cdot\|_{\gamma_\infty}$; it is a closed subspace of $\gamma_\infty(\mathcal{H}, E)$. A celebrated result of Kwapien and Hoffmann-Jorgensen [17, 23] implies that if E does not contain a closed subspace isomorphic to c_0 then $\gamma(\mathcal{H}, E) = \gamma_\infty(\mathcal{H}, E)$.

Since convergence in $\gamma(\mathcal{H}, E)$ implies convergence in $\mathcal{L}(\mathcal{H}, E)$, every operator $R \in \gamma(\mathcal{H}, E)$, being the operator norm limit of a sequence of finite rank operators from \mathcal{H} to E , is compact.

If \mathcal{H} is separable with orthonormal basis $\{h_j\}_{j \geq 1}$, then an operator $R : \mathcal{H} \rightarrow E$ is γ -radonifying if and only if the Gaussian sum $\sum_{j \geq 1} \gamma_j R h_j$ converges in $L^2(\Omega; E)$, and in this situation we have

$$\|R\|_{\gamma(\mathcal{H}, E)}^2 = \mathbb{E} \left\| \sum_{j \geq 1} \gamma_j R h_j \right\|^2.$$

The general case may be reduced to the separable case by observing that for any $R \in \gamma(\mathcal{H}, E)$ there exists a separable closed subspace \mathcal{H}_R of \mathcal{H} such that R vanishes on the orthogonal complement \mathcal{H}_R^\perp .

If $R \in \gamma(\mathcal{H}, E)$ is given and $\{h_j\}_{j \geq 1}$ is an orthonormal basis for \mathcal{H}_R , the sum $\sum_{j \geq 1} \gamma_j R h_j$ defines a centred E -valued Gaussian random variable. Its distribution μ is a centred Gaussian Radon measure on E whose covariance operator equals RR^* . We will refer to μ as the Gaussian measure *associated with* R . In the reverse direction, if Y is a centred E -valued Gaussian random variable with reproducing kernel Hilbert space \mathcal{H} , then \mathcal{H} is separable, the natural inclusion mapping $i : \mathcal{H} \hookrightarrow E$ is γ -radonifying, and we have

$$\|i\|_{\gamma(\mathcal{H}, E)}^2 = \mathbb{E} \|Y\|^2.$$

Below we shall need the following simple continuity result.

PROPOSITION 2.1. *Let (X, d) be a metric space and let $V : X \rightarrow \mathcal{L}(E, F)$ be strongly continuous. Then for all $R \in \gamma(\mathcal{H}, E)$ the function $VR : X \rightarrow \gamma(\mathcal{H}, F)$,*

$$(VR)(\xi) := V(\xi)R, \quad \xi \in X,$$

is continuous.

Proof. Suppose first that R is a finite rank operator, say $R = \sum_{j=1}^k h_j \otimes x_j$ with $\{h_j\}_{j=1}^k \in \mathcal{H}$ orthonormal and $\{x_j\}_{j=1}^k$ a sequence in E . Suppose that $\lim_{n \rightarrow \infty} \xi_n = \xi$ in X . Then

$$\lim_{n \rightarrow \infty} \|V(\xi_n)R - V(\xi)R\|_{\gamma(\mathcal{H}, F)}^2 = \lim_{n \rightarrow \infty} \mathbb{E} \left\| \sum_{j=1}^k \gamma_j(V(\xi_n) - V(\xi))x_j \right\|^2 = 0.$$

The general case follows from the density of the finite rank operators in $\gamma(\mathcal{H}, E)$ and the norm estimate $\|V(\xi)R\|_{\gamma(\mathcal{H}, F)} \leq \|V(\xi)\| \|R\|_{\gamma(\mathcal{H}, E)}$. \square

3. Proof of Theorem 1.1. We start with a brief discussion of stochastic integrals of operator-valued functions. Let H be a Hilbert space and fix $T > 0$. An H -cylindrical Brownian motion, indexed by $[0, T]$ and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is a mapping $W_H : L^2(0, T; H) \rightarrow L^2(\Omega)$ with the following properties:

- for all $h \in L^2(0, T; H)$ the random variable $W_H(h)$ is Gaussian;
- for all $h_1, h_2 \in L^2(0, T; H)$ we have $\mathbb{E}W_H(h_1)W_H(h_2) = \langle h_1, h_2 \rangle$.

Formally, an H -cylindrical Brownian motion can be thought of as a standard Brownian motion taking values in the Hilbert space H . One easily checks that W_H is linear and that for all $h_1, \dots, h_n \in L^2(0, T; H)$ the random variables $W_H(h_1), \dots, W_H(h_n)$ are jointly Gaussian. These random variables are independent if and only if h_1, \dots, h_n are orthogonal in H . For further details see [28, Section 3].

A *finite rank step function* is a function of the form $\sum_{n=1}^N 1_{(a_n, b_n]} \otimes B_n$ where each operator $B_n : H \rightarrow E$ is of finite rank. The stochastic integral with respect to W_H of such a function is defined by setting

$$\int_0^T 1_{(a, b]} \otimes (h \otimes x) dW_H := W_H(1_{(a, b]} \otimes h) \otimes x$$

and this definition is extended by linearity. A function $\Psi : (0, T) \rightarrow \mathcal{L}(H, E)$ is said to be *stochastically integrable* with respect to W_H if there exists a sequence of finite rank step functions $\Psi_n : (0, T) \rightarrow \mathcal{L}(H, E)$ such that:

- (i) for all $h \in H$ we have $\lim_{n \rightarrow \infty} \Psi_n h = \Psi h$ in measure on $(0, T)$;
- (ii) the limit $Y := \lim_{n \rightarrow \infty} \int_0^T \Psi_n dW_H$ exists in probability.

In this situation we write

$$Y = \int_0^T \Psi dW_H$$

and call Y the *stochastic integral* of Ψ with respect to W_H .

As was shown in [32], for finite rank step functions Ψ one has the isometry

$$\mathbb{E} \left\| \int_0^T \Psi dW_H \right\|^2 = \|R_\Psi\|_{\gamma(L^2(0, T; H), E)}^2, \quad (3.1)$$

where $R_\Psi : L^2(0, T; H) \rightarrow E$ is the bounded operator represented by Ψ , i.e.,

$$R_\Psi f = \int_0^T \Psi(t)f(t) dt, \quad f \in L^2(0, T; H). \quad (3.2)$$

As a consequence, a function $\Psi : (0, T) \rightarrow \mathcal{L}(H, E)$ is stochastically integrable on $(0, T)$ with respect to W_H if and only if $\Psi^* x^* \in L^2(0, T; H)$ for all $x^* \in E^*$ and there

exists an operator $R_\Psi \in \gamma(L^2(0, T; H), E)$ such that

$$\langle R_\Psi f, x^* \rangle = \int_0^T [f(t), \Psi^*(t)x^*] dt, \quad x^* \in E^*.$$

The isometry (3.1) extends to this situation. The following simple observation [10, Lemma 2.1] will be used frequently:

PROPOSITION 3.1. *For all $g \in L^2(0, T)$ and $R \in \gamma(H, E)$ the function $gR : t \mapsto g(t)R$ belongs to $\gamma(L^2(0, T; H), E)$ and we have*

$$\|gR\|_{\gamma(L^2(0, T; H), E)} = \|g\|_{L^2(0, T)} \|R\|_{\gamma(H, E)}.$$

For the remainder of this section we fix an E -valued Brownian motion $W = \{W(t)\}_{t \geq 0}$ and $T > 0$. Let H be the reproducing kernel Hilbert space associated with the Gaussian random variable $W(1)$ and let $i : H \hookrightarrow E$ be the natural inclusion mapping. Then W induces an H -cylindrical Brownian motion W_H by putting

$$W_H(f \otimes i^*x^*) := \int_0^T f d\langle W, x^* \rangle, \quad f \in L^2(0, T), \quad x^* \in E^*. \quad (3.3)$$

This motivates us to call a function $\Psi : (0, T) \rightarrow \mathcal{L}(E)$ *stochastically integrable* with respect to W if the function $\Psi \circ i : (0, T) \rightarrow \mathcal{L}(H, E)$ is stochastically integrable with respect to W_H , in which case we put

$$\int_0^T \Psi dW := \int_0^T (\Psi \circ i) dW_H.$$

It is easy to check that for all $S \in \mathcal{L}(E)$ the indicator function $1_{(a, b]} \otimes S$ is stochastically integrable with respect to W and

$$\int_0^T 1_{(a, b]} \otimes S dW = S(W(b) - W(a)).$$

This shows that the definition is consistent with (1.1) and (1.2).

Now let $S = \{S(t)\}_{t \geq 0}$ denote a C_0 -semigroup of bounded linear operators on E , with generator A . We will be interested in the case where the function to be integrated against W_H is one of the following:

$$\Phi(t) := S(t) \circ i, \quad \Phi^{(n)}(t) := \sum_{j=1}^n 1_{I_j^{(n)}}(t) \otimes [S(t_j^{(n)}) \circ i], \quad t \in (0, T).$$

We may define bounded operators $R_{\Phi^{(n)}}$ and R_Φ from $L^2(0, T; H)$ to E by the formula (3.2). Being associated with $\gamma(H, E)$ -valued step functions, the operators $R_{\Phi^{(n)}}$ belong to $\gamma(L^2(0, T; H), E)$ by Proposition 3.1. Concerning the question whether the operator R_Φ is in $\gamma(L^2(0, T; H), E)$ we have the following result [32, Theorem 7.1].

PROPOSITION 3.2. *Let $\Phi(t) = S(t) \circ i$. The following assertions are equivalent:*

- (i) *the operator R_Φ belongs to $\gamma(L^2(0, T; H), E)$;*
- (ii) *the function Φ is stochastically integrable on $(0, T)$ with respect to W_H ;*
- (iii) *for some (all) $x \in E$ the problem (SCP_x) admits a unique solution U_x .*

In this situation, for all $x \in E$ and $t \in [0, T]$ we have

$$U_x(t) = S(t)x + \int_0^t S(t-s) dW(s) = S(t)x + \int_0^t \Phi(t-s) dW_H(s)$$

almost surely. In [32] an example is presented showing even for rank one Brownian motions W in E the equivalent conditions need not always be satisfied for all C_0 -semigroups S on E . The conditions are satisfied, however, if one of the following additional conditions holds:

- (a) E is a type 2 Banach space,
- (b) S restricts to a C_0 -semigroup on H ,
- (c) S is an analytic C_0 -semigroup on E .

We refer to [10, 32] for the easy proofs.

We are now in a position to state the main result of this section. We use the notations introduced above, and let μ and $\mu^{(n)}$ denote the Gaussian measures on E associated with the operators R_Φ and $R_{\Phi^{(n)}}$, respectively.

THEOREM 3.3. *Suppose that the equivalent conditions of Proposition 3.2 are satisfied. The following assertions are equivalent:*

1. $\lim_{n \rightarrow \infty} U_x^{(n)}(T) = U_x(T)$ in $L^p(\Omega; E)$ for some (all) $x \in E$ and some (all) $1 \leq p < \infty$;
2. $\lim_{n \rightarrow \infty} R_{\Phi^{(n)}} = R_\Phi$ in $\gamma(L^2(0, T; H), E)$;
3. $\lim_{n \rightarrow \infty} \mu^{(n)} = \mu$ weakly.

In this situation we have $\lim_{n \rightarrow \infty} U_x^{(n)}(t) = U_x(t)$ in $L^p(\Omega; E)$ for all $x \in E$, $t \in [0, T]$, and $1 \leq p < \infty$, and in fact we have

$$\sup_{0 \leq t \leq T} \mathbb{E} \|U_x^{(n)}(t) - U_x(t)\|^p \leq \sup_{0 \leq t \leq T} \|S^{(n)}(t)x - S(t)x\| + \mathbb{E} \|U^{(n)}(T) - U(T)\|^p,$$

where, as before, $U^{(n)} = U_0^{(n)}$ and $U = U_0$ correspond to the initial value 0.

Proof. We begin by proving the equivalence of (1), (2), (3). Clearly it suffices to consider the initial value $x = 0$.

For a given $1 \leq p < \infty$, a sequence of E -valued centred Gaussian random variables converges in $L^p(\Omega; E)$ if and only if it converges in probability in E . Therefore, if (1) holds for some $1 \leq p < \infty$, then it holds for all $1 \leq p < \infty$.

Taking $p = 2$ in (1) the equivalence (1) \Leftrightarrow (2) follows from the identity (3.1) and the representations (1.1) and (1.2).

Next we claim that $\lim_{n \rightarrow \infty} R_{\Phi^{(n)}}^* x^* = R_\Phi^* x^*$ in $L^2(0, T; H)$ for all $x^* \in E^*$. Once we have shown this, the equivalence (2) \Leftrightarrow (3) follows from [16, Theorem 3.1] (or by using the argument of [34, page 18ff]). To prove the claim we fix $x^* \in E^*$ and note that in $L^2(0, T; H)$ we have

$$R_\Phi^* x^* = i^* S^*(\cdot) x^*, \quad R_{\Phi^{(n)}}^* x^* = \sum_{j=1}^n 1_{I_j^{(n)}}(\cdot) \otimes i^* S^*(t_j^{(n)}) x^*.$$

The inclusion mapping $i : H \hookrightarrow E$ is γ -radonifying and hence compact. As a consequence, the weak*-continuity of $t \mapsto S^*(t)x^*$ implies that $t \mapsto i^* S^*(t)x^* = \Phi^*(t)x^*$ is continuous on $[0, T]$. It follows that $\lim_{n \rightarrow \infty} R_{\Phi^{(n)}}^*(\cdot)x^* = R_\Phi^*(\cdot)x^*$ in $L^\infty(0, T; H)$, and hence in $L^2(0, T; H)$.

The final assertion is an immediate consequence of (1.1), (1.2), and covariance domination [32, Corollary 4.4]. \square

The assertions (1), (2), (3) are equivalent to the validity of a Lie–Trotter product formula for the Ornstein-Uhlenbeck semigroup $\mathcal{P} = \{\mathcal{P}(t)\}_{t \geq 0}$ associated with the problem (SCP_x) , which is defined on the space $C_b(E)$ of all bounded real-valued continuous functions on E by the formula

$$\mathcal{P}(t)f(x) = \mathbb{E}f(U_x(t)), \quad x \in E, \quad t \geq 0,$$

where U_x is the solution of (SCP_x) . In order to explain the precise result, let us denote by $\mathcal{S} = \{\mathcal{S}(t)\}_{t \geq 0}$ and $\mathcal{T} = \{\mathcal{T}(t)\}_{t \geq 0}$ the semigroups on $C_b(E)$ corresponding to the drift term and the diffusion term in (SCP_x) . Thus,

$$\begin{aligned} \mathcal{S}(t)f(x) &= f(S(t)x), \\ \mathcal{T}(t)f(x) &= \mathbb{E}f(x + W(t)), \end{aligned} \quad t \geq 0, \quad x \in E.$$

Each of the semigroups \mathcal{P} , \mathcal{S} and \mathcal{T} is jointly continuous in t and x , uniformly on $[0, T] \times K$ for all compact sets $K \subseteq E$. It was shown in [21] that if condition (3) of Theorem 3.3 holds, then for all $f \in C_b(E)$ we have the Lie–Trotter product formula

$$\mathcal{P}(t)f(x) = \lim_{n \rightarrow \infty} [\mathcal{T}(t/n)\mathcal{S}(t/n)]^n f(x) \quad (3.4)$$

with convergence uniformly on $[0, T] \times K$ for all compact sets $K \subseteq E$. Conversely it follows from the proof of this result that (3.4) with $x = 0$ implies condition (3) of Theorem 3.3. In the same paper it was shown that (3.4) holds if at least one of the next two conditions is satisfied:

- (a) E is isomorphic to a Hilbert space;
- (b) S restricts to a C_0 -semigroup on H .

Thus, either of these conditions implies the convergence $\lim_{n \rightarrow \infty} U_x^{(n)}(t) = U_x(t)$ in $L^p(\Omega; E)$ for all $x \in E$ and $t \in [0, T]$ of the splitting scheme. The proofs in [21] are rather involved. A simple proof for case (b) has been subsequently obtained by Johanna Tikanmäki (personal communication). In Theorems 3.4 and 3.5 below we shall give simple proofs for both cases (a) and (b), based on the Proposition 2.1 and an elementary convergence result for γ -radonifying operators from [30], respectively. Moreover, case (a) is extended to Banach spaces with type 2. Recall that a Banach space is said to have *type* $1 \leq p \leq 2$ if there exists a constant $C \geq 0$ such that for all finite choices $x_1, \dots, x_k \in E$ we have

$$\left(\mathbb{E} \left\| \sum_{j=1}^k \gamma_j x_j \right\|^2 \right)^{\frac{1}{2}} \leq C \left(\sum_{j=1}^k \|x_j\|^p \right)^{\frac{1}{p}}.$$

Hilbert spaces have type 2 and L^p -spaces ($1 \leq p < \infty$) have type $\min\{p, 2\}$. We refer to [1] for more information.

THEOREM 3.4. *If E has type 2, then the equivalent conditions of Proposition 3.2 and Theorem 3.3 hold for every C_0 -semigroup S on E . As a consequence we have*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} \|U^{(n)}(t) - U(t)\|^p = 0, \quad 1 \leq p < \infty.$$

Proof. By Proposition 2.1 we have $\Phi \in C([0, T]; \gamma(H, E))$. This clearly implies that $\lim_{n \rightarrow \infty} \Phi^{(n)} = \Phi$ in $L^\infty(0, T; \gamma(H, E))$, and hence in $L^2(0, T; \gamma(H, E))$. Since

E has type 2, by [33, Lemma 6.1] the mapping $\Psi \mapsto R_\Psi$ defines a continuous inclusion $L^2(0, T; \gamma(H, E)) \hookrightarrow \gamma(L^2(0, T; H), E)$. It follows that $\lim_{n \rightarrow \infty} R_{\Phi^{(n)}} = R_\Phi$ in $\gamma(L^2(0, T; H), E)$. \square

THEOREM 3.5. *If S restricts to a C_0 -semigroup on H , then the equivalent conditions of Proposition 3.2 and Theorem 3.3 hold. As a consequence we have*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} \|U^{(n)}(t) - U(t)\|^p = 0, \quad 1 \leq p < \infty.$$

Proof. Let S_H denote the restricted semigroup on H . From the identity $S(t) \circ i = i \circ S_H(t)$ we have $R_\Phi = i \circ T$ and $R_{\Phi^{(n)}} = i \circ T^{(n)}$, where T and $T^{(n)}$ are the bounded operators from $L^2(0, T; H)$ to H defined by

$$Tf := \int_0^T S_H(t)f(t) dt, \quad T^{(n)}f = \int_0^T \sum_{j=1}^n 1_{I_j^{(n)}}(t) S_H(t_j^{(n)})f(t) dt.$$

Since $\lim_{n \rightarrow \infty} (T^{(n)})^*h = T^*h$ for all $h \in H$ by the strong continuity of the adjoint semigroup S_H^* (see [37]), it follows from [30, Proposition 2.4] that $\lim_{n \rightarrow \infty} R_{\Phi^{(n)}} = R_\Phi$ in $\gamma(L^2(0, T; H), E)$. \square

4. Proof of Theorem 1.2. In this section we shall prove convergence of the splitting scheme under the assumption that the C_0 -semigroup generated by A is analytic; no assumptions on the space E are made. In this situation we are also able to give explicit rates of convergence in suitable interpolation spaces.

We begin with a minor extension of a result due to Kalton and Weis [20]. It enables us to check whether certain $\mathcal{L}(H, E)$ -valued functions define operators belonging to $\gamma(L^2(0, T; H), E)$. We refer to [28, Section 13] for a detailed proof.

PROPOSITION 4.1. *Let $\Phi : (a, b) \rightarrow \gamma(H, E)$ be continuously differentiable with*

$$\int_a^b (s - a)^{\frac{1}{2}} \|\Phi'(s)\|_{\gamma(H, E)} ds < \infty.$$

Define $R_\Phi : L^2(a, b; H) \rightarrow E$ by

$$R_\Phi f := \int_a^b \Phi(t)f(t) dt.$$

Then $R_\Phi \in \gamma(L^2(a, b; H), E)$ and

$$\|R_\Phi\|_{\gamma(L^2(a, b; H), E)} \leq (b - a)^{\frac{1}{2}} \|\Phi(b)\|_{\gamma(H, E)} + \int_a^b (s - a)^{\frac{1}{2}} \|\Phi'(s)\|_{\gamma(H, E)} ds.$$

For $\alpha \geq 0$ and large enough $w \in \mathbb{R}$ we define

$$E_\alpha := \mathcal{D}((w - A)^\alpha),$$

which is known to be independent of the choice of w . It is a Banach space with respect to the norm $\|x\|_{E_\alpha} := \|(w - A)^\alpha x\|$. This norm depends of course on w , but any two such norms are mutually equivalent. In what follows we consider w to be fixed.

We shall also need the extrapolation spaces $E_{-\alpha}$, defined for $\alpha > 0$ as the closure of E with respect to the norm $\|x\|_{E_{-\alpha}} := \|(w - A)^{-\alpha} x\|$. It follows readily from the

definitions that for any two $\alpha, \beta \in \mathbb{R}$ the operator $(w - A)^\alpha$ defines an isomorphism from E_β onto $E_{\beta-\alpha}$.

In the next two remarks we fix $\alpha, \beta \geq 0$ and $i \in \gamma(H, E_\beta)$, and suppose that S is an analytic C_0 -semigroup on E with generator A .

REMARK 4.2. By [35, Theorem 2.6.13(c)] one has, for any $\theta \geq 0$,

$$\|S(t)\|_{\mathcal{L}(E, E_\theta)} \lesssim t^{-\theta} \quad (4.1)$$

with implied constant independent of $t \in [0, T]$. From this and the ideal property for γ -radonifying operators we obtain the following estimate for $\Phi(t) := S(t) \circ i$:

$$\begin{aligned} \|\Phi'(t)\|_{\gamma(H, E_\alpha)} &\leq \|AS(t)\|_{\mathcal{L}(E_\beta, E_\alpha)} \|i\|_{\gamma(H, E_\beta)} \\ &= \|S(t)\|_{\mathcal{L}(E, E_{\alpha+1-\beta})} \|i\|_{\gamma(H, E_\beta)} \lesssim t^{-(\alpha+1-\beta)^+} \|i\|_{\gamma(H, E_\beta)} \end{aligned}$$

where $r^+ := \max\{0, r\}$ for $r \in \mathbb{R}$; the implied constant is independent of $t \in [0, T]$ and $i \in \gamma(H, E_\beta)$. If $\alpha - \beta < \frac{1}{2}$, it then follows from Proposition 4.1 that

$$\|R_\Phi\|_{\gamma(L^2(0, t; H), E_\alpha)} \lesssim t^{\min\{\frac{1}{2}-\alpha+\beta, \frac{3}{2}\}} \|i\|_{\gamma(H, E_\beta)},$$

with implied constant independent of $t \in [0, T]$ and $i \in \gamma(H, E_\beta)$. In particular, taking $\alpha = \beta = 0$ we see that the equivalent conditions of Proposition 3.2 hold.

REMARK 4.3. Suppose that $\delta \in [0, \frac{1}{2})$. Identifying operator-valued functions with the integral operators they induce, we have

$$\begin{aligned} \|s \mapsto s^{-\delta} S(t-s)i\|_{\gamma(L^2(0, t; H), E_\alpha)} \\ \leq \|s \mapsto s^{-\delta} S(t-s)i\|_{\gamma(L^2(0, \frac{t}{2}; H), E_\alpha)} + \|s \mapsto (t-s)^{-\delta} S(s)i\|_{\gamma(L^2(0, \frac{t}{2}; H), E_\alpha)}. \end{aligned}$$

Applying Proposition 4.1 to both terms on the right-hand side, if $\alpha - \beta < \frac{1}{2}$ it follows that

$$[s \mapsto s^{-\delta} S(t-s)i] \in \gamma(L^2(0, t; H), E_\alpha)$$

for all $t \in [0, T]$.

We need to introduce the following terminology. Let E and F be Banach spaces. A family of operators $\mathcal{R} \subseteq \mathcal{L}(E, F)$ is called γ -bounded if there exists a finite constant $C \geq 0$ such that for all finite choices $R_1, \dots, R_N \in \mathcal{R}$ and vectors $x_1, \dots, x_N \in E$ we have

$$\mathbb{E} \left\| \sum_{n=1}^N \gamma_n R_n x_n \right\|^2 \leq C^2 \mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|^2.$$

The least admissible constant C is called the γ -bound of \mathcal{R} , notation $\gamma(\mathcal{R})$. We refer to [6, 9, 22, 40] for examples and more information. In these references the related notion of R -boundedness is discussed; this notion is obtained by replacing the Gaussian random variables by Rademacher variables in the above definition. Any R -bounded set is also γ -bounded, and the two notions are equivalent if E has finite cotype.

We continue with a multiplier result, also due to Kalton and Weis [20]. We refer to [28, Section 5] for a detailed proof.

PROPOSITION 4.4. *Suppose that E and F are Banach spaces and $M : (0, T) \rightarrow \mathcal{L}(E, F)$ is a strongly measurable function (in the sense that $t \mapsto M(t)x$ is strongly*

measurable for every $x \in E$) with γ -bounded range $\mathcal{M} = \{M(t) : t \in (0, T)\}$. Then for every finite rank simple function $\Phi : (0, T) \rightarrow \gamma(H, E)$ the operator $R_{M\Phi}$ belongs to $\gamma_\infty(L^2(0, T; H), F)$ and

$$\|R_{M\Phi}\|_{\gamma_\infty(L^2(0, T; H), F)} \leq \gamma(\mathcal{M}) \|R_\Phi\|_{\gamma(L^2(0, T; H), E)}.$$

As a result, the map $\widetilde{M} : R_\Phi \mapsto R_{M\Phi}$ has a unique extension to a bounded operator

$$\widetilde{M} : \gamma(L^2(0, T; H), E) \rightarrow \gamma_\infty(L^2(0, T; H), F)$$

of norm $\|\widetilde{M}\| \leq \gamma(\mathcal{M})$.

In the applications of this result below it will usually be possible to check that actually we have $R_{M\Phi} \in \gamma(L^2(0, T; H), F)$.

We will also need the following sufficient condition for γ -boundedness, which is a variation of a result of Weis [40, Proposition 2.5].

PROPOSITION 4.5. *Let E and F be Banach spaces, and let $f : (0, T) \rightarrow \mathcal{L}(E, F)$ be an function such that for all $x \in E$ the function $t \mapsto f(t)x$ is continuously differentiable with integrable derivative. Then the set $\mathcal{F} := \{f(t) : t \in (0, T)\}$ is γ -bounded in $\mathcal{L}(E, F)$ and*

$$\gamma(\mathcal{F}) \leq \|f(0+)\| + \|f'\|_1.$$

Here is a simple application:

LEMMA 4.6. *Let the C_0 -semigroup S be analytic on E .*

(1) *For all $0 \leq \alpha < \delta$ and $t \in (0, T]$ the set $\mathcal{S}_{\alpha, \delta, t} = \{s^\delta S(s) : s \in [0, t]\}$ is γ -bounded in $\mathcal{L}(E, E_\alpha)$ and we have*

$$\gamma(\mathcal{S}_{\alpha, \delta, t}) \lesssim t^{\delta - \alpha}, \quad t \in (0, T],$$

with implied constant independent of $t \in (0, T]$.

(2) *For all $0 < \alpha \leq 1$ the set $\mathcal{T}_{\alpha, t} = \{S(s) - I : s \in [0, t]\}$ is γ -bounded in $\mathcal{L}(E_\alpha, E)$ and we have*

$$\gamma(\mathcal{T}_{\alpha, t}) \lesssim t^\alpha, \quad t \in [0, T],$$

with implied constant independent of $t \in [0, T]$.

Proof. For the proof of (1) we refer to [10] or [29, Lemma 10.17]. To prove (2) it will be shown that for any fixed and large enough $w \in \mathbb{R}$ the set

$$\mathcal{T}_{\alpha, t}^w := \{e^{-ws} S(s) - I : s \in [0, t]\}$$

is γ -bounded in $\mathcal{L}(E_\alpha, E)$ with γ -bound $\lesssim t^\alpha$. From this we deduce that $\{S(s) : s \in [0, t]\}$ is γ -bounded in $\mathcal{L}(E_\alpha, E)$ with γ -bound $\lesssim 1$. In view of the identity

$$S(s) - I = (e^{-ws} S(s) - I) + (1 - e^{-ws}) S(s)$$

and noting that $1 - e^{-ws} \lesssim s$, this will prove the assertion of the lemma.

For all $x \in E$ and $0 \leq s \leq t$,

$$e^{-ws} S(s)x - x = \int_0^s e^{-wr} (A - w) S(r)x \, dr.$$

By (4.1) and Proposition 4.5 the set $\mathcal{T}_{\alpha,t}^w$ is γ -bounded in $\mathcal{L}(E_\alpha, E)$ and $\gamma(\mathcal{T}_{\alpha,t}^w) \lesssim \int_0^t s^{\alpha-1} ds \lesssim t^\alpha$. \square

We shall again write $U = U_0$ and $U^{(n)} = U_0^{(n)}$ for the solution of (SCP_0) and its approximations by the splitting scheme.

THEOREM 4.7. *Assume that the semigroup S is analytic on E and that W is a Brownian motion in E_β for some $\beta \geq 0$. Then the equivalent conditions of Proposition 3.2 and Theorem 3.3 hold. Moreover, for all $\alpha \geq 0$ and $0 \leq \theta \leq 1$ such that $\alpha - \beta + \theta < \frac{1}{2}$, and all $t \in [0, T]$ we have*

$$\|R_{\Phi^{(n)}} - R_\Phi\|_{\gamma(L^2(0,t;H), E_\alpha)} \lesssim n^{-\theta} t^{\frac{1}{2} - (\alpha - \beta + \theta)^+}. \quad (4.2)$$

with implied constant independent of $n \geq 1$ and $t \in [0, T]$. As a consequence, for all $1 \leq p < \infty$ the solution U of (SCP_0) satisfies

$$(\mathbb{E}\|U^{(n)}(t) - U(t)\|_{E_\alpha}^p)^{\frac{1}{p}} \lesssim n^{-\theta} t^{\frac{1}{2} - (\alpha - \beta + \theta)^+} \quad (4.3)$$

with implied constant independent of $n \geq 1$ and $t \in [0, T]$.

Proof. The estimate (4.3) follows from (4.2) via Theorem 3.3.

By rescaling time we may assume that $T = 1$. Let α, β, θ be as indicated. We begin by noting that the embedding $i : H \hookrightarrow E$ associated with W belongs to $\gamma(H, E_\beta)$.

Pick $(\alpha - \beta + \theta)^+ < \delta < \frac{1}{2}$. Note that for $0 < s \leq T$ we have $S^{(n)}(s) = S(n^{-1}\lceil ns \rceil)$ and $s \leq n^{-1}\lceil ns \rceil$, so one can write, for all $n \geq 1$,

$$\Phi^{(n)}(s) - \Phi(s) = s^\delta S(s) \circ (S(n^{-1}\lceil ns \rceil - s) - I) \circ s^{-\delta} i. \quad (4.4)$$

Fix $t \in (0, 1]$. By the first part of Lemma 4.6 the set

$$\mathcal{S}_\delta = \{s^\delta S(s) : s \in [0, t]\}$$

is γ -bounded in $\mathcal{L}(E, E_{(\alpha - \beta + \theta)^+})$ (hence in $\mathcal{L}(E, E_{\alpha - \beta + \theta})$, hence in $\mathcal{L}(E_{\beta - \theta}, E_\alpha)$, with the same upper bounds for the γ -bounds, because $S(t)$ commutes with the fractional powers of A) and we have

$$\gamma(\mathcal{S}_\delta) \lesssim t^{\delta - (\alpha - \beta + \theta)^+}. \quad (4.5)$$

By the second part of Lemma 4.6 the set

$$\mathcal{T}_{\theta, \frac{1}{n}} = \{S(s) - I : s \in [0, n^{-1}]\}$$

is γ -bounded in $\mathcal{L}(E_\theta, E)$ (and hence in $\mathcal{L}(E_\beta, E_{\beta - \theta})$, with the same estimate for the γ -boundedness constant), and we have

$$\gamma(\mathcal{T}_{\theta, \frac{1}{n}}) \lesssim n^{-\theta}. \quad (4.6)$$

Using (4.4), Remark 4.3, Proposition 4.4, the identity

$$\|R_{s \mapsto s^{-\delta} i}\|_{\gamma(L^2(0,t;H), E_\beta)} = \|s \mapsto s^{-\delta} \|_{L^2(0,t)} \|i\|_{\gamma(H, E_\beta)} \approx t^{\frac{1}{2} - \delta} \|i\|_{\gamma(H, E_\beta)} \quad (4.7)$$

together with the estimates (4.5), and (4.6), and noting that $n^{-1}\lceil ns \rceil - s \leq n^{-1}$, one obtains

$$\begin{aligned} \|R_{\Phi^{(n)}} - R_\Phi\|_{\gamma(L^2(0,t;H), E_\alpha)} &\leq \gamma(\mathcal{S}_\delta) \gamma(\mathcal{T}_{\theta, \frac{1}{n}}) \|R_{s \mapsto s^{-\delta} i}\|_{\gamma(L^2(0,t;H), E_\beta)} \\ &\lesssim n^{-\theta} t^{\frac{1}{2} - (\alpha - \beta + \theta)^+} \|i\|_{\gamma(H, E_\beta)}. \end{aligned}$$

□

REMARK 4.8. The condition $\alpha - \beta + \theta < \frac{1}{2}$ implies, in view of the restriction $0 \leq \theta \leq 1$, that $\alpha - \beta < \frac{1}{2}$. For $\alpha - \beta < -\frac{1}{2}$, Theorem 4.7 gives a rate of convergence of order n^{-1} , whereas for $-\frac{1}{2} \leq \alpha - \beta < \frac{1}{2}$ we obtain the rate $n^{-\theta}$ for any $0 \leq \theta < \frac{1}{2} - \alpha + \beta$. For $-\frac{1}{2} < \alpha - \beta < \frac{1}{2}$ one can in fact obtain a slightly better rate at the final time T , namely $(\ln \ln n)/n^{(\frac{1}{2}-\alpha+\beta)}$. More precisely, for $n \geq 3$ we have

$$\|R_{\Phi(n)} - R_{\Phi}\|_{\gamma(L^2(0,T;H),E_{\alpha})} \lesssim \frac{\ln \ln n}{n^{\frac{1}{2}-\alpha+\beta}} \quad (4.8)$$

with constants independent of $n \geq 1$.

Once again observe that by scaling we may (and do) assume that $T = 1$. In order to prove (4.8) we first give an estimate for a given time interval $[a, b]$ where $0 < a < b \leq 1$. In that case, for $\delta > \alpha - \beta + 1$ one has

$$\|R_{\Phi(n)} - R_{\Phi}\|_{\gamma(L^2(a,b;H),E_{\alpha})} \lesssim n^{-1} a^{\frac{1}{2}-\delta} b^{\delta-\alpha-\beta+1} \quad (4.9)$$

with implied constant independent of $n \geq 1$ and $0 < a < b \leq 1$. The proof of (4.9) is similar to that of (4.2), the main difference being that we no longer need to ensure the square integrability of $s \mapsto s^{-\delta}$ near $s = 0$ in (4.7). The details are as follows. Fix $n \geq 1$ and $0 < a < b \leq 1$ and pick an arbitrary $\delta > \alpha - \beta + 1$. Then,

$$\|R_{s \mapsto s^{-\delta} i}\|_{\gamma(L^2(a,b;H),E_{\beta})} = \|s \mapsto s^{-\delta}\|_{L^2(a,b)} \|i\|_{\gamma(H,E_{\beta})} \lesssim a^{\frac{1}{2}-\delta} \|i\|_{\gamma(H,E_{\beta})}, \quad (4.10)$$

with implied constant independent of $a \in (0, 1]$ and $b \in (a, 1]$; the last inequality uses that $\delta \geq \frac{1}{2}$. As in the proof of Theorem 4.7 the set $\mathcal{S}_{\frac{1}{n}} := \{S(s) - I : s \in [0, n^{-1}]\}$ is γ -bounded in $\mathcal{L}(E_{\beta}, E_{\beta-1})$, with γ -bound

$$\gamma(\mathcal{S}_{\frac{1}{n}}) \lesssim n^{-1}. \quad (4.11)$$

Finally, since $\delta > \alpha - \beta + 1$, as in the proof of Theorem 4.7 the set $\mathcal{S}_{\delta} = \{s^{\delta} S(s) : s \in [a, b]\}$ is γ -bounded in $\mathcal{L}(E_{\beta-1}, E_{\alpha})$ with

$$\gamma(\mathcal{S}_{\delta}) \lesssim b^{\delta-\alpha-\beta+1}. \quad (4.12)$$

Combining (4.10), (4.11), and (4.12) we obtain

$$\|R_{\Phi(n)} - R_{\Phi}\|_{\gamma(L^2(a,b;H),E_{\alpha})} \lesssim n^{-1} a^{\frac{1}{2}-\delta} b^{\delta-\alpha-\beta+1} \|i\|_{\gamma(H,E_{\beta})}.$$

Returning to the proof of estimate (4.8) we fix an integer $n \geq 3$. Because $\beta - \alpha < \frac{1}{2}$ one can pick $\delta > 0$ such that $1 + \alpha - \beta < \delta \leq \frac{3}{2} + 2(\alpha - \beta)$. For $j = 0, 1, \dots$ define $a_j := n^{-1+2^{-j}}$. Note that $a_0 = 1$ and $\lim_{j \rightarrow \infty} a_j = n^{-1}$. If in (4.9) we take $a = a_j$ and $b = a_{j-1}$ we obtain the estimate

$$\begin{aligned} & \|R_{\Phi(n)} - R_{\Phi}\|_{\gamma(L^2(a_j, a_{j-1}; H), E_{\alpha})} \\ & \lesssim n^{-1+(-1+2^{-j})(\frac{1}{2}-\delta)+(-1+2^{1-j})(\delta-\alpha+\beta-1)} \|i\|_{\gamma(H, E_{\beta})} \\ & = n^{-\frac{1}{2}+\alpha-\beta+2^{-j}(\delta-\frac{3}{2}-2(\alpha-\beta))} \|i\|_{\gamma(H, E_{\beta})} \\ & \leq n^{-\frac{1}{2}+\alpha-\beta} \|i\|_{\gamma(H, E_{\beta})}, \end{aligned}$$

where the last inequality used that $\delta \leq \frac{3}{2} + 2(\alpha - \beta)$. Set $k_n = \lceil (\ln \ln n) / \ln 2 \rceil$, so that $a_{k_n} \leq en^{-1}$. Using this estimate for a_{k_n} , from Theorem 4.7 we obtain, for any choice of $0 \leq \theta < \frac{1}{2} - \alpha + \beta$ (which then satisfies $\theta < 1$),

$$\|R_{\Phi(n)} - R_{\Phi}\|_{\gamma(L^2(0, a_{k_n}; H), E_{\alpha})} \lesssim a_{k_n}^{\frac{1}{2}-\alpha+\beta-\theta} n^{-\theta} \|i\|_{\gamma(H, E_{\beta})} \lesssim n^{-\frac{1}{2}+\alpha-\beta} \|i\|_{\gamma(H, E_{\beta})}.$$

Combining the above one gets

$$\begin{aligned}
& \|R_{\Phi^{(n)}} - R_{\Phi}\|_{\gamma(L^2(0,1;H),E_{\alpha})} \\
& \lesssim \|R_{\Phi^{(n)}} - R_{\Phi}\|_{\gamma(L^2(0,a_{k_n};H),E_{\alpha})} + \sum_{j=1}^{k_N} \|R_{\Phi^{(n)}} - R_{\Phi}\|_{\gamma(L^2(a_j,a_{j-1};H),E_{\alpha})} \\
& \lesssim (1 + \ln \ln n) n^{-\frac{1}{2} + \alpha - \beta} \|i\|_{\gamma(H,E_{\beta})}.
\end{aligned}$$

This gives the estimate (4.8).

Under the assumptions that S is analytic on E and W is a Brownian motion on E , the solution U of (SCP_0) has a version with trajectories in $C^{\gamma}([0,T];E_{\alpha})$ for any $\alpha, \gamma \geq 0$ such that $\alpha + \gamma < \frac{1}{2}$ [10]. The main result of this paper asserts that also the approximating processes $U^{(n)}$ have trajectories in $C^{\gamma}([0,T];E_{\alpha})$ and that the splitting scheme converges with respect to the $C^{\gamma}([0,T];E_{\alpha})$ -norm, with a convergence rate depending on α and γ and the smoothness of the noise.

THEOREM 4.9. *Let S be analytic on E and suppose that W is a Brownian motion in E_{β} for some $\beta \geq 0$. If $\alpha, \theta, \gamma \geq 0$ satisfy $\theta + \gamma < 1$ and $(\alpha - \beta + \theta)^+ + \gamma < \frac{1}{2}$, then for all $1 \leq p < \infty$ the solution U of (SCP_0) satisfies*

$$(\mathbb{E}\|U^{(n)} - U\|_{C^{\gamma}([0,T],E_{\alpha})}^p)^{\frac{1}{p}} \lesssim n^{-\theta},$$

with implied constant independent of $n \geq 1$.

Proof. By scaling we may assume $T = 1$. Put $V^{(n)} := U^{(n)} - U$. Let α, β, γ and θ be as indicated. Without loss of generality we assume that $\gamma > 0$. The main step in the proof is the following claim.

CLAIM 4.10. *There exists a constant C such that for all $n \geq 1$, all $0 \leq s < t \leq 1$ satisfying $t - s < \frac{1}{2n}$ we have*

$$(\mathbb{E}\|V^{(n)}(t) - V^{(n)}(s)\|_{E_{\alpha}}^2)^{\frac{1}{2}} \leq C n^{-\theta} (t - s)^{\gamma}.$$

Proof. Fix $n \geq 1$ and $0 \leq s < t \leq 1$ such that $t - s < \frac{1}{2n}$. Clearly,

$$\begin{aligned}
(\mathbb{E}\|V^{(n)}(t) - V^{(n)}(s)\|_{E_{\alpha}}^2)^{\frac{1}{2}} & \leq \left(\mathbb{E} \left\| \int_s^t \Phi(t-r) - \Phi^{(n)}(t-r) dW(r) \right\|^2 \right)^{\frac{1}{2}} \\
& \quad + \left(\mathbb{E} \left\| \int_0^s \Phi(t-r) - \Phi(s-r) dW(r) \right\|^2 \right)^{\frac{1}{2}} \\
& \quad + \left(\mathbb{E} \left\| \int_0^s \Phi^{(n)}(t-r) - \Phi^{(n)}(s-r) dW(r) \right\|^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.13}$$

For the first term we note that by (3.1) (and the remark following it) and (4.2) one has

$$\begin{aligned}
& \left(\mathbb{E} \left\| \int_s^t \Phi^{(n)}(t-r) - \Phi(t-r) dW_H(r) \right\|_{E_{\alpha}}^2 \right)^{\frac{1}{2}} \\
& = \left(\mathbb{E} \left\| \int_0^{t-s} \Phi^{(n)}(r) - \Phi(r) dW_H(r) \right\|_{E_{\alpha}}^2 \right)^{\frac{1}{2}} \\
& \lesssim n^{-\theta} (t-s)^{\frac{1}{2} - (\alpha - \beta + \theta)^+} \|i\|_{\gamma(H,E_{\beta})} \\
& \leq n^{-\theta} (t-s)^{\gamma} \|i\|_{\gamma(H,E_{\beta})}.
\end{aligned} \tag{4.14}$$

The estimate for the second term is extracted from arguments in [31]; see also [29, Theorem 10.19]. Fix $\eta > 0$ such that $(\alpha - \beta + \theta)^+ + \gamma < \eta < \frac{1}{2}$. Then the set $\{t^\eta S(t) : t \in (0, T)\}$ is γ -bounded in $\mathcal{L}(E, E_{(\alpha-\beta+\theta)^++\gamma})$ (hence in $\mathcal{L}(E, E_{\alpha-\beta+\theta+\gamma})$, hence in $\mathcal{L}(E_{\beta-\theta-\gamma}, E_\alpha)$) by the first part of Lemma 4.6, and therefore

$$\begin{aligned}
& \left(\mathbb{E} \left\| \int_0^s \Phi(t-r) - \Phi(s-r) dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}} \\
&= \left(\mathbb{E} \left\| \int_0^s [(s-r)^\eta S(s-r)] \circ [(s-r)^{-\eta} (S(t-s) - I) \circ i] dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\mathbb{E} \left\| \int_0^s (s-r)^{-\eta} (S(t-s) - I) \circ i dW_H(r) \right\|_{E_{\beta-\theta-\gamma}}^2 \right)^{\frac{1}{2}} \\
&= \left(\int_0^s (s-r)^{-2\eta} dr \right)^{\frac{1}{2}} \| (S(t-s) - I) \circ i \|_{\gamma(H, E_{\beta-\theta-\gamma})} \\
&\lesssim \| S(t-s) - I \|_{\mathcal{L}(E_\beta, E_{\beta-\theta-\gamma})} \| i \|_{\gamma(H, E_\beta)} \\
&\approx \| S(t-s) - I \|_{\mathcal{L}(E_{\gamma+\theta}, E)} \| i \|_{\gamma(H, E_\beta)} \\
&\lesssim (t-s)^{\gamma+\theta} \| i \|_{\gamma(H, E_\beta)} \\
&\lesssim n^{-\theta} (t-s)^\gamma \| i \|_{\gamma(H, E_\beta)}.
\end{aligned} \tag{4.15}$$

To estimate the third term on the right-hand side of (4.13), we first define sets B_0 and B_1 by

$$\begin{aligned}
B_0 &:= \{r \in (0, s) : S^{(n)}(t-r) = S^{(n)}(s-r)\} \\
&= \{r \in (0, s) : \lceil n(t-r) \rceil = \lceil n(s-r) \rceil\}, \\
B_1 &:= \{r \in (0, s) : S^{(n)}(t-r) = S(n^{-1})S^{(n)}(s-r)\} \\
&= \{r \in (0, s) : \lceil n(t-r) \rceil = \lceil n(s-r) \rceil + 1\}.
\end{aligned}$$

Both equalities follow from the identity $S^{(n)}(u) = S(n^{-1}\lceil nu \rceil)$ for $u \in (0, T)$. By definition of B_0 and B_1 one has

$$\begin{aligned}
& \left(\mathbb{E} \left\| \int_0^s \Phi^{(n)}(t-r) - \Phi^{(n)}(s-r) dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E} \left\| \int_{B_0} \Phi^{(n)}(t-r) - \Phi^{(n)}(s-r) dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\mathbb{E} \left\| \int_{B_1} \Phi^{(n)}(t-r) - \Phi^{(n)}(s-r) dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}} \\
&= \left(\mathbb{E} \left\| \int_{B_1} S^{(n)}(s-r)(S(n^{-1}) - I)i dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}},
\end{aligned} \tag{4.16}$$

noting that the integrand of the integral over B_0 vanishes.

Set $\delta := \theta + \gamma$. To estimate the right-hand side, observe that from $\alpha - \beta + \delta < \frac{1}{2}$ we may pick $\eta > 0$ such that $\alpha - \beta + \delta < \eta < \frac{1}{2}$. Using the identity $S^{(n)}(u) = S(n^{-1}\lceil nu \rceil)$ and applying Proposition 4.4 and part (1) of Lemma 4.6, and then using the estimate

$\|S(u) - I\|_{\mathcal{L}(E_\delta, E)} \lesssim u^\delta$ and Proposition 3.1, we obtain

$$\begin{aligned}
& \left(\mathbb{E} \left\| \int_{B_1} S^{(n)}(s-r)(S(n^{-1}) - I) i dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}} \\
& \quad \approx \left(\mathbb{E} \left\| \int_{B_1} (n^{-1} \lceil n(s-r) \rceil)^\eta S(n^{-1} \lceil n(s-r) \rceil) \right. \right. \\
& \quad \quad \times \left. \left. (n^{-1} \lceil n(s-r) \rceil)^{-\eta} (S(n^{-1}) - I) i dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}} \quad (4.17) \\
& \quad \lesssim \left(\mathbb{E} \left\| \int_{B_1} (n^{-1} \lceil n(s-r) \rceil)^{-\eta} (S(n^{-1}) - I) i dW_H(r) \right\|_{E_{\beta-\delta}}^2 \right)^{\frac{1}{2}} \\
& \quad \lesssim n^{-\delta} \|(s - \cdot)^{-\eta}\|_{L^2(B_1)} \|i\|_{\gamma(H, E_\beta)}.
\end{aligned}$$

In order to estimate the $L^2(B_1)$ -norm of the function $f_s(r) := (s-r)^{-\eta}$ we note that $B_1 \subseteq \bigcup_{j=1}^n B_1^{(j)}$, where

$$\begin{aligned}
B_1^{(j)} &= \{r \in (0, s) : s-r \leq jn^{-1} < t-r\} \\
&= \{r \in (0, s) : jn^{-1} - t + s < s-r \leq jn^{-1}\}.
\end{aligned}$$

From this it is easy to see that $|B_1^{(j)}| \leq t-s$ and that for $r \in B_1^{(j)}$ one has

$$(s-r)^{-2\eta} \leq (jn^{-1} - t + s)^{-2\eta} \leq n^{2\eta} (j - \frac{1}{2})^{-2\eta}$$

(the latter inequality following from $t-s < 1/2n$), and therefore

$$\|f_s\|_{L^2(B_1)}^2 = \int_{B_1} |f_s(r)|^2 dr \leq n^{2\eta} |B_1^{(j)}| \sum_{j=1}^n \frac{1}{(j - \frac{1}{2})^{2\eta}} \lesssim n(t-s).$$

As a consequence,

$$\|f_s\|_{L^2(B_1)} \lesssim n^{\frac{1}{2}}(t-s)^{\frac{1}{2}} = n^{\frac{1}{2}}(t-s)^{\frac{1}{2}-\gamma}(t-s)^\gamma \lesssim n^\gamma(t-s)^\gamma. \quad (4.18)$$

Combining the estimates (4.17) and (4.18) and estimating the non-negative powers of s by 1 we find

$$\left(\mathbb{E} \left\| \int_{B_1} S^{(n)}(s-r)(S^{(n)}(t-s) - I) i dW_H(r) \right\|_{E_\alpha}^2 \right)^{\frac{1}{2}} \lesssim n^{-\theta}(t-s)^\gamma \|i\|_{\gamma(H, E_\beta)}. \quad (4.19)$$

Claim 4.10 now follows by combining (4.13), (4.14), (4.15), (4.16) and (4.19). \square

We are now ready to finish the proof of the theorem. By the triangle inequality and Theorem 4.7, for all $0 \leq s, t \leq 1$ we have

$$\begin{aligned}
(\mathbb{E} \|V^{(n)}(t) - V^{(n)}(s)\|_{E_\alpha}^2)^{\frac{1}{2}} &\leq (\mathbb{E} \|U^{(n)}(t) - U(t)\|_{E_\alpha}^2)^{\frac{1}{2}} + (\mathbb{E} \|U^{(n)}(s) - U(s)\|_{E_\alpha}^2)^{\frac{1}{2}} \\
&\lesssim n^{-\delta} \|i\|_{\gamma(H, E_\beta)}.
\end{aligned}$$

Hence if $t-s \geq (2n)^{-1}$ one has

$$(\mathbb{E} \|V^{(n)}(t) - V^{(n)}(s)\|_{E_\alpha}^2)^{\frac{1}{2}} \lesssim n^{-\delta} \|i\|_{\gamma(H, E_\beta)} \lesssim n^{-\theta}(t-s)^\gamma \|i\|_{\gamma(H, E_\beta)}. \quad (4.20)$$

The random variables $V^{(n)}(t)$ being Gaussian, from the claim and (4.20) combined with the Kahane-Khintchine inequalities we deduce that for all $1 \leq q < \infty$ and $0 \leq s < t \leq 1$ one has

$$(\mathbb{E} \|V^{(n)}(t) - V^{(n)}(s)\|_{E_\alpha}^q)^{\frac{1}{q}} \lesssim n^{-\theta} (t-s)^\gamma \|i\|_{\gamma(H, E_\beta)}. \quad (4.21)$$

Now fix any $0 < \gamma' < \gamma$ and take $1/\gamma' < q < \infty$. Then by (4.21) and the Kolmogorov-Chentsov criterion with L^q -moments (see [11, Theorem 5]),

$$\|U^{(n)} - U\|_{L^q(\Omega; C^{\gamma' - \frac{1}{q}}([0, T]; E_\alpha))} \lesssim \|U^{(n)} - U\|_{C^\gamma([0, T]; L^q(\Omega; E_\alpha))} \lesssim n^{-\theta} \|i\|_{\gamma(H, E_\beta)}.$$

This inequality shows that for all $0 < \bar{\gamma} < \gamma$ we have

$$\|U^{(n)} - U\|_{L^q(\Omega; C^{\bar{\gamma}}([0, T]; E_\alpha))} \lesssim n^{-\theta} \|i\|_{\gamma(H, E_\beta)}$$

for all sufficiently large $1 \leq q < \infty$. It is clear that once we know this, this inequality extends to all values $1 \leq q < \infty$. This completes the proof of the theorem (with $\bar{\gamma}$ instead of γ , which obviously suffices). \square

COROLLARY 4.11. *Suppose that S is analytic on E and that W is a Brownian motion in E_β for some $\beta \geq 0$. Let $\alpha, \gamma, \theta \geq 0$ satisfy $\theta + \gamma < 1$ and $(\alpha - \beta + \theta)^+ + \gamma < \frac{1}{2}$. Then for almost all $\omega \in \Omega$ there exists a constant $C(\omega)$ such that the solution U of (SCP_0) satisfies*

$$\|U_x^{(n)}(\cdot, \omega) - U_x(\cdot, \omega)\|_{C^\gamma([0, T]; E_\alpha)} \leq \frac{C(\omega)}{n^\theta} \quad \text{for all } n = 1, 2, \dots$$

Proof. Set

$$\Omega_n := \left\{ \omega \in \Omega : \|U^{(n)}(\cdot, \omega) - U(\cdot, \omega)\|_{C^\gamma([0, T]; E_\alpha)} > \frac{1}{n^\theta} \right\}.$$

Pick $\bar{\theta} > \theta$ in such a way that $0 \leq \alpha - \beta + \gamma + \bar{\theta} < \frac{1}{2}$ and let $p \geq 1$ be so large that $(\bar{\theta} - \theta)p > 1$. By Theorem 4.9, applied with $\bar{\theta}$ instead of θ , and Chebyshev's inequality,

$$\mathbb{P}(\Omega_n) \leq n^{\theta p} \mathbb{E} \|U^{(n)}(\cdot, \omega) - U(\cdot, \omega)\|_{C^\gamma([0, T]; E_\alpha)}^p \leq \frac{C^p}{n^{(\bar{\theta} - \theta)p}}$$

with constant C independent of n . By the choice of p we have $\sum_{n \geq 1} \mathbb{P}(\Omega_n) < \infty$, and therefore by the Borel-Cantelli lemma

$$\mathbb{P}(\{\omega \in \Omega : \omega \in \Omega_n \text{ infinitely often}\}) = 0.$$

For the $\omega \in \Omega$ belonging to this set we have

$$C(\omega) := \sup_{n \geq 1} n^\theta \|U^{(n)}(\cdot, \omega) - U(\cdot, \omega)\|_{C^\gamma([0, T]; E_\alpha)} < \infty.$$

\square

We conclude this section with an application of our results to the stochastic heat equation on the unit interval driven by space-time white noise. This example is merely included as a demonstration how such equations can be handled in the present framework. We don't strive for the greatest possible generality. For instance, as in

[5, 10] the Laplace operator can be replaced by more general second order elliptic operators.

EXAMPLE 4.12. Consider the following stochastic partial differential equation driven by space-time white noise w :

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + \frac{\partial w}{\partial t}(t, x), & x \in [0, 1], t \in [0, T], \\ u(0, x) = 0, & x \in [0, 1], \\ u(t, 0) = u(t, 1) = 0, & t \in [0, T]. \end{cases} \quad (4.22)$$

Following the approach of [10] we put $H := L^2(0, 1)$ and $F := L^q(0, 1)$, where the exponent $q \geq 2$ is to be chosen later on. In order to formulate the problem (4.22) as an abstract stochastic evolution equation of the form

$$\begin{cases} dU(t) = AU(t) dt + dW(t), & t \in [0, T], \\ U(0) = 0, \end{cases} \quad (4.23)$$

where W is a Brownian motion with values in a suitable Banach space E , we fix an arbitrary real number $\rho < -\frac{1}{4}$, to be chosen in a moment, and let $E := F_\rho$ denote the extrapolation space of order $-\rho$ associated with the Dirichlet Laplacian in F . It is shown in [10] (see also [5, Lemma 6.5]) that the identity operator on H extends to a γ -radonifying embedding from H into E . As a result, the H -cylindrical Brownian motion W_H canonically associated with w (see (3.3)) may be identified with a Brownian motion W in E . Furthermore the extrapolated Dirichlet Laplacian, henceforth denoted by A , satisfies the assumptions of Theorem 4.9 in E .

Let U be the solution of (4.23) in E . By definition, we shall regard U as the solution of (4.22). Suppose now that we are given real numbers $\gamma, \delta, \theta \geq 0$ satisfy

$$\gamma + \delta + \theta < \frac{1}{4}.$$

This ensures that one can choose $\alpha \geq 0$ and $\rho < -\frac{1}{4}$ in such a way that $\alpha + \rho > \delta$ and $\alpha + \gamma + \theta < \frac{1}{2}$. By Theorem 4.9 (with $\beta = 0$), for all $1 \leq p < \infty$ the splitting scheme associated with problem (4.23) satisfies

$$(\mathbb{E} \|U^{(n)} - U\|_{C^\gamma([0, T], E_\alpha)}^p)^{\frac{1}{p}} \lesssim n^{-\theta}.$$

Putting $\eta := \alpha + \rho$ we have $E_\alpha = (F_\rho)_\alpha = F_\eta$, and this space embeds into F since $\eta > \delta \geq 0$.

Choose $q \geq 2$ so large that $2\delta + \frac{1}{q} < 2\eta$. We have

$$F_\eta = H_0^{2\eta, q}(0, 1) = \{f \in H^{2\eta, q}(0, 1) : f(0) = f(1) = 0\}$$

with equivalent norms. By the Sobolev embedding theorem,

$$H^{2\eta, q}(0, 1) \hookrightarrow C^{2\delta}[0, 1]$$

with continuous inclusion. Here $C^{2\delta}[0, 1]$ is the space of all Hölder continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ of exponent 2δ . We denote $C_0^{2\delta}[0, 1] = \{f \in C^{2\delta}[0, 1] : f(0) = f(1) = 0\}$. Putting things together we obtain a continuous inclusion

$$F_\eta \hookrightarrow C_0^{2\delta}[0, 1].$$

We have proved the following theorem (cf. Example 1.4).

THEOREM 4.13. *For all $0 \leq \delta < \frac{1}{4}$ the stochastic heat equation (4.22) admits a solution U in $C_0^{2\delta}[0, 1]$, and for all $\gamma, \theta \geq 0$ satisfying $\gamma + \delta + \theta < \frac{1}{4}$ we have*

$$\left(\mathbb{E}\|U^{(n)} - U\|_{C^\gamma([0, T], C_0^{2\delta}[0, 1])}^p\right)^{\frac{1}{p}} \lesssim n^{-\theta}.$$

By Corollary 4.11, we also obtain almost sure convergence with respect to the norm of $C^\gamma([0, T], C_0^{2\delta}[0, 1])$ with rate $1/n^\theta$.

5. A counterexample for convergence. We shall now present an example of a C_0 -semigroup S on a Banach space E and an E -valued Brownian motion W such that the problem (SCP₀) admits a solution with continuous trajectories whilst the associated splitting scheme fails to converge. Although the actual construction is somewhat involved, the semigroup in this example is a translation semigroup on a suitable vector-valued Lebesgue space. Such semigroups occur naturally in the context of stochastic delay equations.

We take $E = L^q(0, 1; \ell^p)$, with $1 \leq p < 2$ and $q \geq 2$, and consider the E -valued Brownian motion $W_f = w \otimes f$, where w is a standard real-valued Brownian motion and $f \in E$ is a fixed element. With this notation a function $\Psi : (0, 1) \rightarrow \mathcal{L}(E)$ is stochastically integrable with respect to W_f if and only if $\Psi f : (0, 1) \rightarrow E$ is stochastically integrable with respect to w , in which case we have

$$\int_0^1 \Psi dW_f = \int_0^1 \Psi f dw.$$

Let $1 \leq p < 2$ and $u > \frac{2}{p}$ be fixed. For $k = 1, 2, \dots$ and $j = 0, \dots, 2^{k-1} - 1$ define the intervals $I_{k,j} = (\frac{2j+1}{2^k}, \frac{2j+1}{2^k} + 2^{-uk}]$. As in particular $u > 1$, for all $k = 1, 2, \dots$ the intervals $I_{k,i}$ and $I_{k,j}$ are disjoint for $i \neq j$. Let $0 < r < 1 - \frac{p}{2}$ and denote the basic sequence of unit vectors in ℓ^p by $\{e_n\}_{n \geq 1}$. Inspired by [38, Example 3.2] we define $f \in L^\infty(\mathbb{R}; \ell^p)$ by

$$f(t) := \sum_{k=1}^{\infty} \sum_{j=0}^{2^{k-1}-1} 2^{-\frac{r}{p}k} 1_{I_{k,j}}(t) e_{2^{k-1}+j}.$$

Observe that $f(t) = 0$ for $t \in \mathbb{R} \setminus (0, 1)$ and f is well-defined: because $I_{k,j}$ and $I_{k,i}$ are disjoint for $i \neq j$ one has, for any $t \in (0, 1)$,

$$\|f(t)\|_{\ell^p}^p \leq \sum_{k=1}^{\infty} 2^{-rk} < \infty.$$

For a given interval $I = (a, b]$, $0 \leq a < b < \infty$, we write $\Delta w_I := w(b) - w(a)$.

CLAIM 5.1. *The function f is stochastically integrable on $(0, 1)$ and*

$$\begin{aligned} \int_0^1 f(t) dw(t) &= \sum_{n=1}^{\infty} \int_0^1 \langle f(t), e_n^* \rangle e_n dw(t) \\ &= \sum_{k=1}^{\infty} \sum_{j=0}^{2^{k-1}-1} 2^{-\frac{r}{p}k} \Delta w_{I_{k,j}} e_{2^{k-1}+j}, \end{aligned} \tag{5.1}$$

where $\{e_n^*\}_{n \geq 1}$ is the basic sequence of unit vectors in $\ell^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. We shall deduce this from [32, Theorem 2.3, (3) \Rightarrow (1)]. Define the ℓ^p -valued Gaussian random variable

$$X := \sum_{k=1}^{\infty} \sum_{j=0}^{2^{k-1}-1} 2^{-\frac{r}{p}k} \Delta w_{I_{k,j}} e_{2^{k-1}+j}.$$

This sum converges absolutely in $L^p(\Omega; \ell^p)$. Indeed, let γ denote a standard Gaussian random variable. Then by Fubini's theorem one has

$$\begin{aligned} \mathbb{E} \left\| \sum_{k=1}^{\infty} \sum_{j=0}^{2^{k-1}-1} 2^{-\frac{r}{p}k} \Delta w_{I_{k,j}} e_{2^{k-1}+j} \right\|_{\ell^p}^p &= \sum_{k=1}^{\infty} \sum_{j=0}^{2^{k-1}-1} 2^{-rk} 2^{-\frac{p}{2}k} \mathbb{E} |\gamma|^p \\ &= \sum_{k=1}^{\infty} 2^{k(1-r-\frac{p}{2})-1} \mathbb{E} |\gamma|^p < \infty. \end{aligned}$$

By the Kahane-Khintchine inequalities, the sum defining X converges absolutely in $L^q(\Omega; \ell^p)$ for all $1 \leq q < \infty$.

For any linear combination $a^* = \sum_{n=1}^N a_n e_n^* \in \ell^{p'}$ one easily checks that

$$\langle X, a^* \rangle = \int_0^1 \langle f(t), a^* \rangle dw(t).$$

Hence by [32, Theorem 2.3], f is stochastically integrable and (5.1) holds. \square

By similar reasoning (or an application of [32, Corollary 2.7]), for all $s \in \mathbb{R}$ the function $t \mapsto f(t+s)$ is stochastically integrable on $(0, 1)$ and

$$\int_0^1 f(t+s) dw(t) = \sum_{n=1}^{\infty} \int_0^1 \langle f(t+s), e_n^* \rangle e_n dw(t).$$

Let $q \geq 1$ and let $\{S(t)\}_{t \in \mathbb{R}}$ be the left-shift group on $L^q(\mathbb{R}; \ell^p)$ defined by

$$(S(t)g)(s) = g(t+s), \quad s, t \in \mathbb{R}, \quad g \in L^q(\mathbb{R}; \ell^p).$$

CLAIM 5.2. *For any $q \geq 1$ the $L^q(\mathbb{R}; \ell^p)$ -valued function $t \mapsto S(t)f$ is stochastically integrable on $(0, 1)$ and*

$$\left(\int_0^1 S(t)f dw(t) \right)(s) = \int_0^1 f(t+s) dw(t)$$

for almost all $s \in \mathbb{R}$ almost surely.

Proof. For $s \notin (-1, 1)$ the function $t \mapsto f(t+s)$ is identically 0 on $(0, 1)$, and for $s \in (-1, 1)$ we have

$$\mathbb{E} \left\| \int_0^1 f(t+s) dw(t) \right\|_{\ell^p}^q \leq \mathbb{E} \left\| \int_0^1 f(t) dw(t) \right\|_{\ell^p}^q.$$

As a consequence, $L^q(\Omega; \ell^p)$ -valued function $s \mapsto \int_0^1 f(s+t) dw(t)$ defines an element of $L^q(\mathbb{R}; L^q(\Omega; \ell^p))$. Under the natural isometry $L^q(\mathbb{R}; L^q(\Omega; \ell^p)) \simeq L^q(\Omega; L^q(\mathbb{R}; \ell^p))$

we may identify this function with an element $Y \in L^q(\Omega; L^q(\mathbb{R}; \ell^p))$. To establish the claim, with an appeal to [32, Theorem 2.3] it suffices to check that for all $a^* \in \ell^{p'}$ and Borel sets $A \in \mathcal{B}(\mathbb{R})$ we have

$$\int_0^1 \langle S(t)f, 1_A \otimes a^* \rangle dw(t) = \langle Y, 1_A \otimes a^* \rangle.$$

By writing out both sides, this identity is seen to be an immediate consequence of the stochastic Fubini theorem (see, e.g., [32, Theorem 3.3]). \square

In the same way one sees that for $t \geq 0$ the stochastic integrals $\int_0^t S(-s)f dw(s)$ are well-defined. Because the process $t \mapsto \int_0^t S(-s)f dw(s)$ is a martingale having a continuous version by Doob's maximal inequality, we also know that the convolution process

$$U(t) := \int_0^t S(t-s)f dw(s) = S(t) \int_0^t S(-s)f dw(s)$$

has a continuous version. However, as we shall see, the splitting scheme for U fails to converge.

For $n \geq 1$ define

$$S^{(n)} := \sum_{k=1}^n 1_{(\frac{k-1}{n}, \frac{k}{n}]} \otimes S\left(\frac{k}{n}\right).$$

Observe that for any $s, t \in \mathbb{R}$

$$(S^{(n)}(t)f)(s) = \sum_{k=1}^n 1_{(\frac{k-1}{n}, \frac{k}{n}]}(t) f\left(\frac{k}{n} + s\right). \quad (5.2)$$

Similarly to the above one checks that

$$\left(\int_0^1 S^{(n)}(t)f dw(t) \right)(s) = \sum_{k=1}^n f\left(\frac{k}{n} + s\right) \left[w\left(\frac{k}{n}\right) - w\left(\frac{k-1}{n}\right) \right]$$

for almost all $s \in \mathbb{R}$ almost surely.

The clue to this example is that for n fixed and $s \in (0, 2^{-un}]$ the function $t \mapsto (S^{(2^n)}(t)f)(s)$ always 'picks up' the values of f at the left parts of the dyadic intervals where f is defined to be non-zero. Thus for these values of s the function $t \mapsto (S^{(2^n)}(t)f)(s)$ it is nowhere zero and its stochastic integral blows up as $n \rightarrow \infty$. We shall make this precise. Our aim is to prove that for certain values of $q > 2$ (to be determined later on) one has

$$\mathbb{E} \left\| \int_0^1 S^{(2^n)}(t)f dw(t) \right\|_{L^q(\mathbb{R}; \ell^p)}^p \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (5.3)$$

By Minkowski's inequality we have, for any $n \geq 1$ and $q \geq p$,

$$\begin{aligned} & \left[\mathbb{E} \left\| \int_0^1 S^{(2^n)}(t)f dw(t) \right\|_{L^q(\mathbb{R}; \ell^p)}^p \right]^{\frac{1}{p}} \\ & \geq \left[\int_{\mathbb{R}} \left(\mathbb{E} \left\| \left(\int_0^1 S^{(2^n)}(t)f dw(t) \right)(s) \right\|_{\ell^p}^p \right)^{\frac{q}{p}} ds \right]^{\frac{1}{q}}. \end{aligned}$$

Now fix $n \geq 1$. For any $1 \leq k \leq n$ and any $j = 0, \dots, 2^{k-1} - 1$ there exists a unique $1 \leq i_{k,j} \leq 2^n - 1$ such that $\frac{i_{k,j}}{2^n} = \frac{2j+1}{2^k}$. Now observe that by definition of f one has for $s \in (0, 2^{-un}]$ that

$$\left\langle f\left(\frac{i_{k,j}}{2^n} + s\right), e_{2^{k-1}+j}^* \right\rangle = 2^{-k\frac{r}{p}}.$$

Using this and representation (5.2) one obtains that for $s \in (0, 2^{-un}]$, $1 \leq k \leq n$, $j = 0, \dots, 2^{k-1} - 1$, and any $t \in (\frac{i_{k,j}-1}{2^n}, \frac{i_{k,j}}{2^n}] =: I_{k,j}^n$,

$$\langle (S^{(2^n)}(t)f)(s), e_{2^{k-1}+j}^* \rangle = 2^{-k\frac{r}{p}}. \quad (5.4)$$

To prove (5.3), we now estimate

$$\begin{aligned} & \int_{\mathbb{R}} \left(\mathbb{E} \left\| \left(\int_0^1 S^{(2^n)}(t)f \, dw(t) \right)(s) \right\|_{\ell^p}^p \right)^{\frac{q}{p}} ds \\ & \geq \int_0^{2^{-un}} \left(\sum_{k=1}^n \sum_{j=0}^{2^{k-1}-1} \mathbb{E} \left| \int_0^1 \langle (S^{(2^n)}(t)f)(s), e_{2^{k-1}+j}^* \rangle dw(t) \right|^p \right)^{\frac{q}{p}} ds \\ & \geq \int_0^{2^{-un}} \left(\sum_{k=1}^n \sum_{j=0}^{2^{k-1}-1} 2^{-kr} \mathbb{E} |\Delta w_{I_{k,j}^n}|^p \right)^{\frac{q}{p}} ds \\ & = \int_0^{2^{-un}} \left(\sum_{k=1}^n 2^{k-1} 2^{-kr} 2^{-n\frac{r}{2}} \mathbb{E} |\gamma|^p \right)^{\frac{q}{p}} ds \\ & \geq 2^{-un-1} 2^{n(1-r-\frac{r}{2})\frac{q}{p}} (\mathbb{E} |\gamma|^p)^{\frac{q}{p}}, \end{aligned}$$

where in the second inequality we use (5.4) and γ denotes a standard Gaussian random variable. Thus if $-u + (1 - r - \frac{r}{2})\frac{q}{p} > 0$, that is, if $q > up/(1 - r - \frac{r}{2})$ (recall that $r < 1 - \frac{r}{2}$), the left-hand side expression diverges as $n \rightarrow \infty$.

Acknowledgments. We thank Ben Goldys, Arnulf Jentzen, Markus Kunze, and Mark Veraar for helpful comments and for suggesting several improvements. We also thank the anonymous referee for pointing out various references.

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